

Consensus Computations over Random Graph Processes*

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Abstract

Distributed consensus processing over random graphs and with randomized node dynamics is considered. At each time step k , every node independently updates its state with a weighted average of its neighbors' states or stick with its current state. The choice is a Bernoulli trial with success probability P_k . The random graph processes, defining the time-varying neighbor sets, are specified over the set of all possible graphs with the given node set. Connectivity-independent and arc-independent graph processes are introduced to capture the fundamental influence of random graphs on the consensus convergence. Necessary and sufficient conditions are presented on the success probability sequence $\{P_k\}$ for the network to reach a global almost sure consensus. For connectivity-independent graphs, we show that $\sum_k P_k^{n-1} = \infty$ is a sufficient condition for almost sure consensus, where n is the number of nodes. For arc-independent graphs, we show that $\sum_k P_k = \infty$ is a sharp threshold, i.e., the consensus probability is zero for almost all initial conditions when the sum converges, while it is one for all initial conditions when the sum diverges. Convergence rates are established by lower and upper bounds of the ϵ -computation time. The results add to the understanding of the interplay between random graphs, random computations, and convergence probability for distributed information processing.

Keywords: consensus algorithms, random graphs, randomized algorithms, threshold

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1 Introduction

In recent years, there has been considerable research interest on distributed algorithms for information exchange, estimation, and computation over networks. Such algorithms have a variety of potential applications in sensor networks, peer-to-peer networks, wireless networks and networked control systems. Targeting the design of simple decentralized algorithms for computation or estimation, where each node exchanges information only with neighbors, distributed averaging serves as a primitive towards more sophisticated information processing algorithms, e.g., [21, 22, 24, 37, 13].

The investigation of the consensus problem has a long history, which can be traced back to 1950s on the study of ergodicity of non-homogeneous Markov chains described as the product of stochastic matrices [16, 15]. The computation of such products have a natural distributed structure, and was used to model consensus, or distributed averaging, algorithms in computer science [18, 19], engineering [20, 28], and social science [17, 43, 44]. Deterministic consensus algorithms have been extensively studied for both time-invariant and time-varying communication graphs in the literature, with efforts typically devoted to finding proper connectivity conditions that can ensure a desired collective convergence [29, 20, 28, 26, 27]. In addition, researchers were also interested in the design of weighted averaging algorithms to reach a faster consensus, or reach a consensus with asynchronous computations [13, 12, 37, 38, 14].

The underlying communication graph over which the consensus algorithm is carried out may be random. The Erdős–Rényi graph, usually denoted as $\mathcal{G}(n, p)$, is a classical model, in which each edge exists randomly, and independently of other edges with probability $p \in (0, 1]$ over a network with n nodes. In [30], the authors studied linear consensus dynamics with communication graphs defined as a sequence of independent, identically distributed (i.i.d.) Erdős–Rényi random graphs, and almost sure convergence was shown. Then in [31], the analysis was generalized to directed Erdős–Rényi graphs. Mean-square performance for consensus algorithms over i.i.d. random graphs was studied in [33], and the influence of random packet drop was investigated in [34]. In [23], a necessary and sufficient condition was presented on almost sure asymptotic consensus for independent random graphs. In [40], the authors studied distributed average consensus in sensor networks with quantized data and random link failures. The independence of the switching communication graphs may then not hold. In [36], the communication graph was described as a finite-state Markov chain where each graph corresponds to one state of the chain, and almost sure consensus was concluded by investigating the connectivity of the

closed positive recurrent sets of the Markovian random graph. In [39], convergence to consensus was studied under more general linear consensus algorithms, where the random update and control matrices were determined by possibly non-stationary stochastic matrix processes coupled with disturbances.

Classical random graph theory suggests that many important properties of graphs appear suddenly from a probabilistic point of view [7]. To be precise, there is usually a function, called a threshold, of a certain parameter, such that almost every graph has a given property if this parameter grows faster than the function. Otherwise almost every graph fails to have the property. Examples of properties considered in the literature include connectivity, k -connectivity, and the Hamiltonian property. The phenomenon is called a zero-one law. It has been shown that $\ln n/n$ is such a threshold function for the connectivity of $\mathcal{G}(n, p)$ in the classical work by Erdős and Rényi [8]. Similar threshold functions were shown for random geometric graphs nodes randomly distributed in a unit square and their connections determined by distances [9, 10]. In [11], a threshold function was established on the θ -connectivity of random graphs defined by a nearest neighbor rule.

Naturally, one may wonder, would there be any threshold condition which leads to a similar zero-one law with respect to collective convergence of consensus dynamics over random graphs? Although various results have been established to guarantee probabilistic consensus [30, 31, 33, 34, 39], the literature still lacks a model and analysis for consensus dynamics over random graphs which can accurately describe the fundamental influence of graph processes and whether such a zero-one law could arise.

To this end, in this paper, we study a randomized consensus algorithm over random graphs. At each time step k , every node independently updates its state with a weighted average of its neighbors' states or stick with its current state. The choice is a Bernoulli trial with success probability P_k . The random graph processes are modeled as a sequence of random variables which take value from the set of all possible graphs with the given node set. We introduce connectivity-independence and arc-independence for the considered random graph processes. For connectivity-independent graphs, we show that $\sum_k P_k^{n-1} = \infty$ is a sufficient condition for almost sure consensus, where n is the number of nodes. For arc-independent graphs, we show that $\sum_k P_k = \infty$ is a sharp threshold, i.e., the consensus probability is zero for almost all initial conditions when the sum converges, while it is one for all initial conditions when the sum diverges. In other words, a zero-one law is established for the presented randomized consensus

processing over arc-independent graphs, and we see that the success probability of the node updates defines the parameter for the corresponding threshold function. Hence, also consensus computations over random graph processes show the sudden transition similar to Erdős–Rényi graphs, and other models in the literature.

1.1 Problem Definition

Consider a network with node set $\mathcal{V} = \{1, 2, \dots, n\}$. A directed graph (digraph), $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ consists of a node set \mathcal{V} and an arc set \mathcal{E} , where each element $e = (i, j) \in \mathcal{E}$ is an ordered pair of two different nodes i and j in \mathcal{V} . [3]. There are $2^{n(n-1)}$ different digraphs with node set \mathcal{V} . We label these graphs from 1 to $2^{n(n-1)}$ by an arbitrary order. In the following, we will identify an integer in $\Omega = \{1, \dots, 2^{n(n-1)}\}$ with the corresponding graph in this order. So Ω represents the graph set.

We model the node interactions as the following graph process.

Definition 1 (*Graph Process*) *The graph process over time is a sequence of random variables that take value in Ω , denoted by $\mathcal{G}_k(\omega) = (\mathcal{V}, \mathcal{E}_k(\omega))$, $k = 0, 1, \dots$.*

When there is no possibility for confusion, we write $\mathcal{G}_k(\omega)$ as \mathcal{G}_k . We call node j a *neighbor* of i at time k if there is an arc from j to i in \mathcal{G}_k . Each node is supposed to always be a neighbor of itself. Denote the random set $\mathcal{N}_i(k) = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}_k\} \cup \{i\}$ as the neighbor set of node i at time k . Let $\{P_k\}_0^\infty$ be a given deterministic sequence with $0 \leq P_k < 1$ for all k . Independent of the graph process, node states, and other nodes, the updating rule of node i is as follows:

$$x_i(k+1) = \begin{cases} \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) x_j(k), & \text{with probability } P_k \\ x_i(k), & \text{with probability } 1 - P_k \end{cases} \quad (1)$$

where $a_{ij}(k)$ denotes the weight of arc (j, i) . We use the following weights rule as our standing assumption, cf., [22, 26, 27, 42].

Assumption (*Weights Rule*) There exists a constant $\eta > 0$ such that

- (i) $\sum_{j \in \mathcal{N}_i(k)} a_{ij}(k) = 1$ for all i and k ;
- (ii) $a_{ij}(k) \geq \eta$ for all i, j and k .

Remark 1 *The randomized information processing (1) describes the possible unwillingness or inability of a node to update, even if it received information from other nodes. For instance, in the opinion dynamics of social networks, forceful belief exchanges may happen randomly by Bernoulli decisions for individuals so that misinformation may be spread [43]. From an engineering viewpoint, in wireless communication nodes may be asleep or broken randomly due to the unpredictability of the environment and the unreliability of the networked communication [33, 40].*

Our interest is in the convergence of the randomized gossip consensus algorithm and the time it takes for the network to reach a consensus. Let

$$x(k; x^0) = (x_1(k; x_1(0)) \dots x_n(k; x_n(0)))^T \in \mathbb{R}^n$$

be the random sequence driven by the randomized algorithm (1) for initial condition $x^0 = (x_1(0) \dots x_n(0))^T$. When it is clear from the context, we will identify $x(k; x^0)$ with $x(k)$. Denote

$$H(k) \doteq \max_{i=1, \dots, n} x_i(k), \quad h(k) \doteq \min_{i=1, \dots, n} x_i(k)$$

as the maximal and minimal states among all nodes, respectively, and define $\mathcal{H}(k) \doteq H(k) - h(k)$ as a consensus measure. We introduce the following definition.

Definition 2 *Global almost sure (a.s.) consensus is achieved for (1) if*

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0\right) = 1 \tag{2}$$

for all $x^0 \in \mathbb{R}^n$. Moreover, for any $0 < \epsilon < 1$, the ϵ -computation time is

$$T_{\text{com}}(\epsilon) \doteq \sup_{x^0 \in \mathbb{R}^n} \inf \left\{ k : \mathbf{P}\left(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\right) \leq \epsilon \right\}. \tag{3}$$

1.2 Main Results

We first present a conclusion on the impossibility of consensus.

Theorem 1 *Global a.s. consensus cannot be achieved for (1) if $\sum_{k=0}^{\infty} P_k < \infty$. In fact, a lower bound for $T_{\text{com}}(\epsilon)$ is given by*

$$T_{\text{com}}(\epsilon) \geq \sup \left\{ k : \sum_{i=0}^{k-1} \log(1 - P_i)^{-1} \leq \frac{\log \epsilon^{-1}}{n} \right\}.$$

Note that, Theorem 1 holds for all possible random graph processes. If we add a simple self-confidence assumption, this impossibility claim can be improved as follows.

Theorem 2 *Suppose there exists a constant a_* such that $a_{ii}(k) \geq a_* > 1/2$ for all i and k . Then*

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0\right) = 0 \quad (4)$$

for almost all initial conditions if $\sum_{k=0}^{\infty} P_k < \infty$.

Recall that a digraph \mathcal{G} is said to be *quasi-strongly connected* if \mathcal{G} has a root [4]. We introduce the following definition.

Definition 3 *Let $\{\mathcal{G}_k\}_0^\infty$ be a random graph process. Then $\{\mathcal{G}_k\}_0^\infty$ is connectivity-independent if the events $A_k \doteq \{\mathcal{G}_k \text{ is quasi-strongly connected}\}$, $k = 0, 1, \dots$, are independent.*

For connectivity-independent graphs, the following conclusion holds.

Theorem 3 *Suppose $\{\mathcal{G}_k\}_0^\infty$ is connectivity-independent and there exists a constant $0 < q < 1$ such that $\mathbf{P}(\mathcal{G}_k \text{ is quasi-strongly connected}) \geq q$ for all k . Assume in addition that $P_{k+1} \leq P_k$. Then global a.s. consensus is achieved for (1) if $\sum_{s=0}^{\infty} P_k^{n-1} = \infty$. Moreover, an upper bound for $T_{\text{com}}(\epsilon)$ is given by*

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{i=0}^{M-1} \log \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot P_{(i+1)(n-1)^2}^{n-1} \right)^{-1} \geq \log \epsilon^{-2} \right\} \times (n-1)^2.$$

We believe that the convergence condition given in Theorem 3 is quite tight since the probability that all the arcs in Algorithm (1) are active at time k is P_k^n , while one single inactive arc is enough to break the connectivity of the graph. Nevertheless, connectivity is a global property of a graph, and indeed it does not rely on any specific arc. The next definition is on the independence of the existence of the arcs in the graph process.

Definition 4 *Let $\{\mathcal{G}_k\}_0^\infty$ be a random graph process. Suppose $\mathcal{G}^* = (\mathcal{V}, \mathcal{E}^*)$ is a given deterministic graph. Then $\{\mathcal{G}_k\}_0^\infty$ is arc-independent with respect to \mathcal{G}^* if the events $B_{k,\tau} \doteq \{(i_\tau, j_\tau) \in \mathcal{G}_k\}$, $(i_\tau, j_\tau) \in \mathcal{E}^*, k = 0, 1, \dots$, are independent. We call \mathcal{G}^* a basic graph of the random graph process.*

For arc-independent graphs, we have the following result.

Theorem 4 Suppose $\{\mathcal{G}_k\}_0^\infty$ is arc-independent with respect to a quasi-strongly connected graph $\mathcal{G}^* = (\mathcal{V}, \mathcal{E}^*)$, and there exists a constant $0 < \theta_0 < 1$ such that $\mathbf{P}((i, j) \in \mathcal{E}_k) \geq \theta_0$ for all k and $(i, j) \in \mathcal{E}^*$. Then (1) achieves global a.s. consensus if and only if $\sum_{k=0}^\infty P_k = \infty$. Moreover, we have

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ k : \sum_{i=0}^{k-1} (1 - (1 - P_i)^n) \geq \frac{(n-1)|\mathcal{E}^*|}{\log A} \log(A\epsilon^2/n) \right\} \quad (5)$$

where $A = 1 - \left(\frac{\eta\theta_0}{n}\right)^{(n-1)|\mathcal{E}^*|}$ and $|\mathcal{E}^*|$ represents the number of elements in \mathcal{E}^* .

The convergence conditions on the success probability sequence $\{P_k\}$ given in Theorems 3 and 4 are similar to the convergence conditions in stochastic optimization, recursive algorithms, adaptive control etc. For example, they are consistent with the decreasing gain condition in the study of stochastic approximation, to guarantee almost sure convergence of adaptive algorithms [6]. Theorems 2 and 4 combined suggest that $\sum_{k=0}^\infty P_k = \infty$ is a sharp threshold for (1) to reach a.s. consensus under the assumption of arc-independent graphs and self-confident updates: the consensus probability is zero for almost all initial conditions when the sum converges, while it is one for all initial conditions when the sum diverges.

1.3 Paper Organization

The rest of the paper is organized as follows. In Section 2, we prove the results on the impossibilities of consensus convergence of Theorems 1 and 2. Section 3 presents convergence analysis for connectivity-independent graphs. In fact, we study some generalized cases which only rely on the independence and connectivity of joint graphs on different time intervals. The concept of joint connectivity has been widely studied in the literature for deterministic consensus algorithms [21, 22, 29, 26, 27]. We investigate directed, bidirectional and acyclic graphs and convergence conditions are given in each case. The proof of Theorem 3 is obtained as a special case. In Section 4, we turn to arc-independent graphs. The proof of Theorem 4 is carried out using a stochastic matrix argument. Finally some concluding remarks are given in Section 5.

2 Impossibilities for Consensus Convergence

This section focuses on the proof of Theorems 1 and 2. The following lemma is well-known.

Lemma 1 Suppose $0 \leq b_k < 1$ for all k . Then $\sum_{k=0}^\infty b_k = \infty$ if and only if $\prod_{k=0}^\infty (1 - b_k) = 0$.

2.1 Proof of Theorem 1

From Algorithm (1), if $\sum_{k=0}^{\infty} P_k < \infty$, we have

$$\mathbf{P}\left(x_i(k+1) = x_i(k), k = 0, 1, \dots\right) \geq \prod_{k=0}^{\infty} (1 - P_k) \doteq r_0,$$

where $0 < r_0 < 1$ according to Lemma 1. Then it is straightforward to see that the impossibility claim of Theorem 1 holds.

Next, we define a scalar random variable $\varpi(k) = \mathcal{H}(k+1)/\mathcal{H}(k)$ when $\mathcal{H}(k) > 0$, and $\varpi(k) = 1$ when $\mathcal{H}(k) = 0$. Obviously, $h(k)$ is non-decreasing, and $H(k)$ is non-increasing. Thus, it always holds that $\varpi(k) \leq 1$. We see from the considered algorithm that

$$\mathbf{P}\left(\varpi(k) = 1\right) \geq (1 - P_k)^n. \quad (6)$$

As a result, we obtain

$$\mathbf{P}\left(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\right) \geq \mathbf{P}\left(\varpi(j) = 1, j = 0, \dots, k-1\right) \geq \prod_{j=0}^{k-1} (1 - P_j)^n, \quad (7)$$

and then the lower bound for the ϵ -computation given in Theorem 1 can be easily obtained. The proof of Theorem 1 is completed.

2.2 Proof of Theorem 2

In order to prove Theorem 2, we need the following lemma.

Lemma 2 *Suppose there exists a constant a_* such that $a_{ii}(k) \geq a_* > 1/2$ for all i and k . Then for all $k \geq 0$, we have*

$$\mathcal{H}(k+1) \geq (2a_* - 1)\mathcal{H}(k).$$

Proof. Suppose $x_m(k) = h(k)$ for some $m \in \mathcal{V}$. Then we have

$$\begin{aligned} \sum_{j \in \mathcal{N}_m(k)} a_{mj}(k)x_j(k) &= a_{mm}(k)x_m(k) + \sum_{j \in \mathcal{N}_m(k) \setminus \{m\}} a_{mj}(k)x_j(k) \\ &\leq a_{mm}(k)h(k) + (1 - a_{mm}(k))H(k) \\ &\leq a_*h(k) + (1 - a_*)H(k), \end{aligned}$$

which implies

$$h(k+1) \leq a_*h(k) + (1 - a_*)H(k). \quad (8)$$

A symmetric argument leads to

$$H(k+1) \geq (1 - a_*)h(k) + a_*H(k). \quad (9)$$

Based on (8) and (9), we obtain

$$\begin{aligned} \mathcal{H}(k+1) &= H(k+1) - h(k+1) \\ &\geq (1 - a_*)h(k) + a_*H(k) - [a_*h(k) + (1 - a_*)H(k)] \\ &\geq (2a_* - 1)\mathcal{H}(k). \end{aligned} \quad (10)$$

The desired conclusion follows. \square

Noting the fact that Lemma 2 holds for all graphs, we see that

$$\mathbf{P}\left(2a_* - 1 \leq \varpi(k) \leq 1, k \geq 0\right) = 1 \quad (11)$$

and

$$\begin{aligned} \mathbf{P}\left(\varpi(k) < 1\right) &\leq \mathbf{P}\left(\text{at least one node takes averaging at time } k\right) \\ &= 1 - (1 - P_k)^n \end{aligned} \quad (12)$$

where $\varpi(k)$ was defined in the proof of Theorem 1.

Next, by Lemma 1, it is not hard to establish

$$\begin{aligned} \sum_{k=0}^{\infty} P_k < \infty &\iff \prod_{k=0}^{\infty} (1 - P_k) > 0 \\ &\iff \prod_{k=0}^{\infty} (1 - P_k)^n > 0 \\ &\iff \sum_{k=0}^{\infty} (1 - (1 - P_k)^n) < \infty, \end{aligned} \quad (13)$$

where the last equivalence is obtained by taking $b_k = 1 - (1 - P_k)^n$ in Lemma 1.

Therefore, if $\sum_{k=0}^{\infty} P_k < \infty$, applying the First Borel-Cantelli Lemma [5] on (12), it follows immediately that

$$\mathbf{P}\left(\varpi(k) < 1 \text{ for infinitely many } k\right) = 0. \quad (14)$$

Furthermore, noticing that (11) implies consensus can only be achieved via infinite time steps, we eventually have

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} \mathcal{H}(k) = 0 \text{ for } \mathcal{H}(0) > 0\right) \leq \mathbf{P}\left(\varpi(k) < 1 \text{ for infinitely many } k\right) = 0. \quad (15)$$

Since $\{x(0) : \mathcal{H}(0) = 0\}$ has zero measure in \mathbb{R}^n , Theorem 2 follows and this ends the proof.

3 Connectivity-independent Graphs

In this section, we present the convergence analysis for connectivity-independent graphs. We are going to study some general cases relying on the joint graphs only.

Joint connectivity has been widely studied in the literature on consensus seeking [21, 22]. The joint graph of \mathcal{G}_k on time interval $[k_1, k_2]$ for $0 \leq k_1 \leq k_2 \leq \infty$ is denoted by $\mathcal{G}([k_1, k_2]) = (\mathcal{V}, \bigcup_{k \in [k_1, k_2]} \mathcal{E}_k)$. We introduce the following definition of joint connectivity for the random graph process.

Definition 5 Let $\{\mathcal{G}_k\}_0^\infty$ be a random graph process.

(i) We call $\{\mathcal{G}_k\}_0^\infty$ stochastically uniformly quasi-strongly connected, if there exist two constants $B \geq 1$ and $0 < q < 1$ such that $\{\mathcal{G}([mB, (m+1)B - 1])\}_0^\infty$ is connectivity-independent and

$$\mathbf{P}\left(\mathcal{G}([mB, (m+1)B - 1]) \text{ is quasi-strongly connected}\right) \geq q, \quad m = 0, 1, \dots$$

(ii) We call $\{\mathcal{G}_k\}_0^\infty$ stochastically infinitely quasi-strongly connected, if there exist a (deterministic) sequence $0 = C_0 < \dots < C_m < \dots$ and a constant $0 < q < 1$ such that $\{\mathcal{G}([C_m, C_{m+1}])\}_{m=0}^\infty$ is connectivity-independent and

$$\mathbf{P}\left(\mathcal{G}([C_m, C_{m+1}]) \text{ is quasi-strongly connected}\right) \geq q, \quad m = 0, 1, \dots$$

Next, in Subsection 3.1, we study stochastically uniformly jointly quasi-strongly connected graph processes, and the proof of Theorem 3 is obtained as a special case. Then in Subsections 3.2 and 3.3, two special cases, bidirectional and acyclic graph processes, are investigated, respectively.

3.1 Bounded Joint Connections

The following result is for consensus convergence on stochastically uniformly quasi-strongly connected graphs.

Proposition 1 Suppose $\{\mathcal{G}_k\}_0^\infty$ is stochastically uniformly quasi-strongly connected with $B \geq 1$ and $q > 0$. Then (1) achieves global a.s. consensus if $\sum_{s=0}^\infty \bar{P}_s = \infty$, where

$$\bar{P}_s = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_j \in [s(n-1)^2 B, (s+1)(n-1)^2 B), j = 1, \dots, n-1 \right\}.$$

Moreover, we have

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{i=0}^{M-1} \log \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{P}_i \right)^{-1} \geq \log \epsilon^{-2} \right\} \times (n-1)^2 B,$$

where η is the constant defined in the weights rule.

In order to prove Proposition 1, we first establish a lemma characterizing a useful property of stochastically uniformly jointly quasi-strongly connected graphs.

Lemma 3 *Assume that \mathcal{G}_k is stochastically uniformly quasi-strongly connected. Then for any $s = 0, 1, \dots$, we have*

$$\mathbf{P}\left(\exists i_0 \in \mathcal{V} \text{ and } \tau_1 < \dots < \tau_{n-1} \in [s(n-1)^2, (s+1)(n-1)^2) \right. \\ \left. \text{s.t. } i_0 \text{ is a center of } \mathcal{G}([\tau_j B, (\tau_j + 1)B - 1]) \text{ for all } j = 1, \dots, n-1\right) \geq q^{(n-1)^2}.$$

Proof. Since \mathcal{G}_k is stochastically uniformly quasi-strongly connected, the probability that each graph $\mathcal{G}([\tau B, (\tau + 1)B - 1])$ for $\tau = s(n-1)^2, \dots, (s+1)(n-1)^2 - 1$, has a center is no less than $q^{(n-1)^2}$. We count a time whenever there is a center node in $\mathcal{G}([\tau B, (\tau + 1)B - 1])$ for $\tau = s(n-1)^2, \dots, (s+1)(n-1)^2 - 1$. These $(n-1)^2$ graphs will lead to at least $(n-1)^2$ counts. However, the total number of the nodes is n . Thus, at least one node is counted more than $(n-2)$ times. The conclusion follows. \square .

We are now ready to prove Proposition 1.

Proof of Proposition 1: Denote $k_s = s(n-1)^2 B$ for $s \geq 0$. Let i_0 be the center node defined in Lemma 3 such that the probability that i_0 is a center of $\mathcal{G}([\tau_j B, (\tau_j + 1)B - 1])$ for $j = 1, \dots, n-1$ with $k_s \leq \tau_j B \leq k_{s+1} - 1$ is no less than $q^{(n-1)^2}$. Assume that

$$x_{i_0}(k_s) \leq \frac{1}{2}h(k_s) + \frac{1}{2}H(k_s). \quad (16)$$

We divide the rest of the proof into four steps.

Step 1. In this step, we bound $x_{i_0}(k)$. With the weights rule of the standing assumption, we have

$$\begin{aligned} \sum_{j \in \mathcal{N}_{i_0}(k_s)} a_{i_0 j}(k_s) x_j(k_s) &= a_{i_0 i_0}(k_s) x_{i_0}(k_s) + \sum_{j \in \mathcal{N}_{i_0}(k_s) \setminus \{i_0\}} a_{i_0 j}(k_s) x_j(k_s) \\ &\leq a_{i_0 i_0}(k_s) \left(\frac{1}{2}h(k_s) + \frac{1}{2}H(k_s) \right) + (1 - a_{i_0 i_0}(k_s)) H(k_s) \\ &\leq \frac{\eta}{2}h(k_s) + \left(1 - \frac{\eta}{2}\right)H(k_s). \end{aligned} \quad (17)$$

Thus, since $\eta < 1$, no matter node i_0 takes averaging or sticks to its current state, we obtain

$$x_{i_0}(k_s + 1) \leq \frac{\eta}{2}h(k_s) + \left(1 - \frac{\eta}{2}\right)H(k_s). \quad (18)$$

Continuing we know that for any $\varrho = 0, 1, \dots$,

$$x_{i_0}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + (1 - \frac{\eta^\varrho}{2})H(k_s). \quad (19)$$

Step 2. Conditioned that i_0 is a center of $\mathcal{G}([\tau_1 B, (\tau_1 + 1)B - 1])$, there will be $i_1 \in \mathcal{V}$, different from i_0 , and a time instance $\hat{k}_1 \in [\tau_1 B, (\tau_1 + 1)B - 1]$, such that $(i_0, i_1) \in \mathcal{E}_{\hat{k}_1}$. Suppose $\hat{k}_1 = k_s + \varsigma$. In this step, we bound $x_{i_1}(k)$.

If i_1 takes an averaging update at time step \hat{k}_1 , with (19), we have

$$\begin{aligned} x_{i_1}(k_s + \varsigma + 1) &= a_{i_1 i_0}(k_s + \varsigma)x_{i_0}(k_s + \varsigma) + \sum_{j \in \mathcal{N}_{i_1}(k_s + \varsigma) \setminus \{i_0\}} a_{i_1 j}(k_s + \varsigma)x_j(k_s + \varsigma) \\ &\leq \eta \left[\frac{\eta^\varsigma}{2}h(k_s) + (1 - \frac{\eta^\varsigma}{2})H(k_s) \right] + (1 - \eta)H(k_s) \\ &= \frac{\eta^{\varsigma+1}}{2}h(k_s) + (1 - \frac{\eta^{\varsigma+1}}{2})H(k_s), \end{aligned} \quad (20)$$

which leads to that for any $\varrho = (\tau_1 + 1)B - k_s, \dots$,

$$x_{i_1}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + (1 - \frac{\eta^\varrho}{2})H(k_s). \quad (21)$$

Therefore, we conclude from (19) and (21) that

$$\mathbf{P}\left(x_{i_l}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + (1 - \frac{\eta^\varrho}{2})H(k_s), \ l = 0, 1; \varrho \geq (\tau_1 + 1)B - k_s\right) \geq q^{(n-1)^2} \min_{k \in [\tau_1 B, (\tau_1 + 1)B - 1]} P_k.$$

Step 3. We proceed the analysis on time interval $[\tau_2 B, (\tau_2 + 1)B - 1]$. Similarly, $i_2 \neq i_0, i_1$ can be found such that

$$\begin{aligned} \mathbf{P}\left(x_{i_l}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + (1 - \frac{\eta^\varrho}{2})H(k_s), \ l = 0, 1, 2; \varrho \geq (\tau_2 + 1)B - k_s\right) \\ \geq q^{(n-1)^2} \min_{\alpha_1, \alpha_2 \in [\tau_1 B, (\tau_2 + 1)B - 1]} P_{\alpha_1} P_{\alpha_2}. \end{aligned}$$

Repeating on time intervals $[\tau_j B, (\tau_j + 1)B - 1]$ for $j = 3, \dots, n - 1$, bounds for i_3, \dots, i_{n-1} can be established as

$$\mathbf{P}\left(x_{i_l}(k_s + \varrho) \leq \frac{\eta^\varrho}{2}h(k_s) + (1 - \frac{\eta^\varrho}{2})H(k_s), \ l = 0, \dots, n - 1; \varrho = (\tau_{n-1} + 1)B - k_s, \dots\right) \geq \bar{P}_s q^{(n-1)^2},$$

with

$$\bar{P}_s = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_j \in [s(n-1)^2 B, (s+1)(n-1)^2 B], j = 1, \dots, n-1 \right\}.$$

Thus, we have

$$\mathbf{P}\left(H(k_{s+1}) \leq \frac{\eta^{(n-1)^2}}{2}h(k_s) + (1 - \frac{\eta^{(n-1)^2}}{2})H(k_s)\right) \geq \bar{P}_s q^{(n-1)^2}, \quad (22)$$

which immediately implies

$$\mathbf{P}\left(\mathcal{H}(k_{s+1}) \leq \left(1 - \frac{\eta^{(n-1)^2}}{2}\right)\mathcal{H}(k_s)\right) \geq \bar{P}_s q^{(n-1)^2}. \quad (23)$$

Note that from a symmetric analysis by establishing the lower bound of $h(k_{s+1})$, it is easy to show that (23) also holds conditioned

$$x_{i_0}(k_s) > \frac{1}{2}h(k_s) + \frac{1}{2}H(k_s). \quad (24)$$

Step 4. With (23), we have

$$\mathbf{E}[\mathcal{H}(k_{s+1})] \leq \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{P}_s\right) \mathbf{E}[\mathcal{H}(k_s)], \quad (25)$$

which leads to

$$\mathbf{E}[\mathcal{H}(k_M)] \leq \prod_{s=0}^{M-1} \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{P}_s\right) \mathcal{H}(0) \quad (26)$$

for all $M \geq 1$ according to the connectivity-independence. Thus, it follows from Lemma 1 that

$$\lim_{M \rightarrow \infty} \mathbf{E}[\mathcal{H}(k_M)] = 0, \quad (27)$$

which yields $\lim_{k \rightarrow \infty} \mathbf{E}[\mathcal{H}(k)] = 0$ since $\mathcal{H}(k)$ is non-increasing. Using Fatou's Lemma, we further obtain

$$0 \leq \mathbf{E}\left[\lim_{k \rightarrow \infty} \mathcal{H}(k)\right] \leq \lim_{k \rightarrow \infty} \mathbf{E}[\mathcal{H}(k)] = 0. \quad (28)$$

Therefore, the convergence claim of the conclusion holds because (28) implies

$$\mathbf{P}\left(\lim_{k \rightarrow +\infty} \mathcal{H}(k) = 0\right) = 1.$$

Applying Markov's Inequality to (26) leads to

$$\mathbf{P}\left(\frac{\mathcal{H}(k_M)}{\mathcal{H}(0)} \geq \epsilon\right) \leq \frac{1}{\epsilon} \cdot \frac{\mathbf{E}[\mathcal{H}(k_M)]}{\mathcal{H}(0)} \leq \frac{1}{\epsilon} \prod_{s=0}^{M-1} \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{P}_s\right). \quad (29)$$

Consequently, we have

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ M : \sum_{s=0}^{M-1} \log \left(1 - \frac{(q\eta)^{(n-1)^2}}{2} \cdot \bar{P}_s\right)^{-1} \geq \log \epsilon^{-2} \right\} \times (n-1)^2 B. \quad (30)$$

The desired conclusion follows. \square

Suppose $P_{k+1} \leq P_k$ for all k . Then it is not hard to see that $\sum_{s=0}^{\infty} \bar{P}_s = \infty$ if and only if $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$. Therefore, the following corollary follows immediately from Proposition 1.

Corollary 1 *Suppose $\{\mathcal{G}_k\}_0^{\infty}$ is stochastically uniformly quasi-strongly connected and $P_{k+1} \leq P_k$ for all k . Then (1) achieves global a.s. consensus if $\sum_{k=0}^{\infty} P_k^{n-1} = \infty$.*

Now we see that Theorem 3 holds as a special case of Corollary 1 with $B = 1$ in the joint connectivity definition.

3.2 Bidirectional Connections

A diagraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is *bidirectional* if for any two nodes i and j , $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. In this subsection, we study bidirectional graphs.

Note that we do not impose an upper bound for the length of the intervals $[C_m, C_{m+1})$ in the definition of stochastically infinitely quasi-strong connectivity, which makes an essential difference from the bounded joint connections. The main result for bidirectional communication under stochastically joint connectivity is stated as follows. Note that in this case quasi-strongly connected graphs is simplified as connected bidirectional graphs.

Proposition 2 *Suppose $\mathbf{P}(\mathcal{G}_k \text{ is bidirectional}, k \geq 0) = 1$. Assume that $\{\mathcal{G}_k\}_0^\infty$ is stochastically infinitely connected. Then (1) achieves global a.s. consensus if $\sum_{s=0}^\infty \hat{P}_s = \infty$, where*

$$\hat{P}_s = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_j \in [C_{s(n-1)}, C_{(s+1)(n-1)}), j = 1, \dots, n-1 \right\}.$$

and also

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ C_{s(n-1)} : \sum_{i=0}^{s-1} \log \left(1 - (q\eta)^{(n-1)} \cdot \hat{P}_i \right)^{-1} \geq \log \epsilon^{-2} \right\}.$$

Proof. We will prove the conclusion by showing $\mathbf{P}(\lim_{k \rightarrow +\infty} \mathcal{H}(k) = 0) = 1$. Take a node $i_0 \in \mathcal{V}$ with $x_{i_0}(C_0) = h(C_0)$.

Define $t_1 = \inf_{k \geq C_0} \{i_0 \text{ has at least one neighbor in } \mathcal{G}_k\}$ and $\mathcal{V}_1 = \{j : j \text{ is a neighbor of } i_0 \text{ at time } t_1\}$. Then we have $\mathbf{P}(t_1 < C_1) \geq q$ because $\{\mathcal{G}_k\}_0^\infty$ is stochastically infinitely connected. Noticing that

$$\begin{aligned} \sum_{j \in \mathcal{N}_{i_0}(t_1)} a_{i_0 j}(t_1) x_j(t_1) &= a_{i_0 i_0}(t_1) x_{i_0}(t_1) + \sum_{j \in \mathcal{N}_{i_0}(t_1) \setminus \{i_0\}} a_{i_0 j}(t_1) x_j(t_1) \\ &\leq a_{i_0 i_0}(t_1) h(C_0) + (1 - a_{i_0 i_0}(t_1)) H(C_0) \\ &\leq \eta h(C_0) + (1 - \eta) H(C_0). \end{aligned} \tag{31}$$

we have

$$\mathbf{P}\left(x_{i_0}(t_1 + 1) \leq \eta h(C_0) + (1 - \eta) H(C_0)\right) = 1. \tag{32}$$

Similarly, for $i_1 \in \mathcal{V}_1$, we have

$$\begin{aligned} \sum_{j \in \mathcal{N}_{i_1}(t_1)} a_{i_1 j}(t_1) x_j(t_1) &= a_{i_1 i_0}(t_1) x_{i_0}(t_1) + \sum_{j \in \mathcal{N}_{i_1}(t_1) \setminus \{i_0\}} a_{i_1 j}(t_1) x_j(t_1) \\ &\leq a_{i_1 i_0}(t_1) h(C_0) + (1 - a_{i_1 i_0}(t_1)) H(C_0) \\ &\leq \eta h(C_0) + (1 - \eta) H(C_0), \end{aligned} \tag{33}$$

which implies

$$\mathbf{P}\left(x_{i_1}(t_1 + 1) \leq \eta h(C_0) + (1 - \eta)H(C_0)\right) \geq P_{t_1}. \quad (34)$$

Thus, we conclude from (32) and (34) that

$$\mathbf{P}\left(x_l(t_1 + 1) \leq \eta h(C_0) + (1 - \eta)H(C_0), l \in \{i_0\} \cup \mathcal{V}_1\right) \geq P_{t_1}^{|\mathcal{V}_1|}. \quad (35)$$

Furthermore, we define $t_2 = \inf_{k \geq k_1} \{\text{at least one other node is connected to } \{i_0\} \cup \mathcal{V}_1 \text{ in } \mathcal{G}_k\}$ and $\mathcal{V}_2 = \{j \in \mathcal{V} \setminus (\{i_0\} \cup \mathcal{V}_1) : j \text{ is connected to } \{i_0\} \cup \mathcal{V}_1 \text{ at } k = t_2\}$. Again we have $\mathbf{P}(t_2 < C_2 | t_1 < C_1) \geq q$. Similar analysis leads to

$$\mathbf{P}\left(x_l(t_2 + 1) \leq \eta^2 h(C_0) + (1 - \eta^2)H(C_0), l \in \{i_0\} \cup \mathcal{V}_1 \cup \mathcal{V}_2\right) \geq P_{t_1}^{|\mathcal{V}_1|} P_{t_2}^{|\mathcal{V}_2|}. \quad (36)$$

Continuing the analysis, t_3, \dots, t_{μ_0} and $\mathcal{V}_3, \dots, \mathcal{V}_{\mu_0}$ can be defined until $\mathcal{V} = \{i_0\} \cup (\bigcup_{l=1}^{\mu_0} \mathcal{V}_l)$ for some $\mu_0 \leq n - 1$, and eventually

$$\mathbf{P}\left(x_l(t_{\mu_0} + 1) \leq \eta^{\mu_0} h(C_0) + (1 - \eta^{\mu_0})H(C_0), l \in \mathcal{V}\right) \geq \prod_{l=1}^{\mu_0} P_{t_l}^{|\mathcal{V}_l|}, \quad (37)$$

which yields

$$\mathbf{P}\left(\mathcal{H}(t_{\mu_0} + 1) \leq (1 - \eta^{\mu_0})\mathcal{H}(C_0)\right) \geq \prod_{l=1}^{\mu_0} P_{t_l}^{|\mathcal{V}_l|}. \quad (38)$$

Therefore, because $\mu_0 \leq n - 1$ and $\mathbf{P}(t_{\mu_0} < C_{n-1}) \geq q^{n-1}$, we have

$$\mathbf{P}\left(\mathcal{H}(C_{n-1}) \leq (1 - \eta^{n-1})\mathcal{H}(C_0)\right) \geq \hat{P}_0 q^{n-1} \quad (39)$$

with $\hat{P}_0 = \inf_{\alpha_1 < \dots < \alpha_{n-1}} \left\{ \prod_{l=1}^{n-1} P_{\alpha_l} : \alpha_j \in [C_0, C_{n-1}), j = 1, \dots, n-1 \right\}$.

Bounds for $\mathcal{H}(C_{s(n-1)})$ can be similarly obtained for $s = 1, 2, \dots$ as

$$\mathbf{P}\left(\mathcal{H}(C_{(s+1)(n-1)}) \leq (1 - \eta^{n-1})\mathcal{H}(C_{s(n-1)})\right) \geq \hat{P}_s q^{n-1}. \quad (40)$$

Therefore, by the same argument as in the proof of Proposition 1, global a.s. consensus holds and

$$T_{\text{com}}(\epsilon) \leq \inf \left\{ C_{s(n-1)} : \sum_{i=0}^{s-1} \log \left(1 - (q\eta)^{(n-1)} \cdot \hat{P}_i \right)^{-1} \geq \log \epsilon^{-2} \right\}. \quad (41)$$

The proof is completed. \square

The following corollary follows directly from Proposition 2.

Corollary 2 Suppose $\mathbf{P}(\mathcal{G}_k \text{ is bidirectional}, k \geq 0) = 1$ and $P_{k+1} \leq P_k$ for all k . Suppose \mathcal{G}_k is stochastically infinitely connected. Then (1) achieves global a.s. consensus if $\sum_{s=0}^{\infty} P_{C_{s(n-1)}}^{n-1} = \infty$.

3.3 Acyclic Connections

A digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is acyclic if it contains no cycle. This subsection focuses on acyclic communication.

Let \mathcal{G} be an acyclic, quasi-strongly connected digraph. Then it is not hard to see that \mathcal{G} has one and only one center. Denote this center as v_0 and let $v_0 \rightarrow j$ be a path from v_0 to j in \mathcal{G} . Let $|v_0 \rightarrow j|$ represent the length of $v_0 \rightarrow j$ as the number of arcs in this path. We now define a function on \mathcal{G} by $\bar{h}(v_0) = 0$ and $\bar{h}(j) = \max\{|v_0 \rightarrow j| : v_0 \rightarrow j \text{ is a path in } \mathcal{G}\}$ for any $j \neq v_0$. Let $d_* = \max_{i \in \mathcal{V}} \bar{h}(i)$. Then we establish the following lemma indicating that this function \bar{h} is surjective from \mathcal{V} to $\{0, \dots, d_*\}$.

Lemma 4 *Let \mathcal{G} be an acyclic, quasi-strongly connected digraph. Then $\bar{h}^{-1}(m) = \{i : \bar{h}(i) = m\}$ is nonempty for any $m = 0, \dots, d_*$.*

Proof. The conclusion holds for $m = 0$ trivially.

Let us prove the conclusion for $m = 1$ by contradiction. Assume that $\bar{h}^{-1}(1) = \emptyset$. Then we have $m_0 \doteq \inf_{i \neq v_0} \bar{h}(i) > 1$. Take a node j_0 with $\bar{h}(j_0) = m_0$. There exists a (simple) path $v_0 \rightarrow j_0$ in \mathcal{G} with length $m_0 > 1$. Let v_* be the node for which arc (v_*, j_0) is included in $v_0 \rightarrow j_0$. According to the definition of m_0 , we have $\bar{h}(v_*) \geq m_0$. Suppose $v_0 \rightarrow v_*$ is a path with length $\bar{h}(v_*)$. Note that, j_0 cannot be included in $v_0 \rightarrow v_*$ because then it would generate a cycle $j_0 \rightarrow v_* \rightarrow j_0$. Consequently, another path $v_0 \rightarrow v_* \rightarrow j_0$ is obtained whose length is $\bar{h}(v_*) + 1 > m_0$. This contradicts the selection rule of j_0 . Therefore, the conclusion holds for $m = 1$.

Next, we construct another graph $\bar{\mathcal{G}}$ from \mathcal{G} by viewing node set $\{v_0\} \cup \bar{h}^{-1}(1)$ as a single node in the new graph without changing the arcs. We see that $\bar{h}^{-1}(2)$ of \mathcal{G} is exactly the same as the node set $\bar{h}^{-1}(1)$ of $\bar{\mathcal{G}}$, while the latter is nonempty via previous analysis. Continuing the argument, the conclusion follows. \square

Here comes our main result for acyclic graphs.

Proposition 3 *Assume that $\mathbf{P}(\mathcal{G}([0, \infty)) \text{ is acyclic}) = 1$ and $\{\mathcal{G}_k\}_0^\infty$ is stochastically infinitely quasi-strongly connected. Then (1) achieves global a.s. consensus if $\sum_{s=0}^\infty \tilde{P}_s = \infty$, where $\tilde{P}_s = \inf_{C_s \leq \alpha < C_{s+1}} P_\alpha$, $s = 0, 1, \dots$.*

Proof. Let v_0 be the unique center node of $\mathcal{G}([0, \infty))$. Based on Lemma 4, $\bar{\mathcal{V}}_i = \bar{h}^{-1}(i)$ for $i = 0, \dots, \bar{h}_0$ can be defined with $\bar{\mathcal{V}}_0 = \{v_0\}$ and $\mathcal{V} = \bigcup_{i=0}^{\bar{h}_0} \bar{\mathcal{V}}_i$.

Obviously we have $\mathbf{P}(x_{v_0}(k) = x_{v_0}(0), k \geq 0) = 1$ because with probability one, v_0 has no neighbor except itself for all k . We turn to $\bar{\mathcal{V}}_1$.

Claim. $\mathbf{P}\left(\lim_{k \rightarrow \infty} |x_l(k) - x_{v_0}(0)| = 0\right) = 1$ for all $l \in \bar{\mathcal{V}}_1$.

Take $v_1 \in \bar{\mathcal{V}}_1$. With probability one, v_0 is the only neighbor of v_1 excluding itself in \mathcal{G}_k for all k . Define $t_1 = \inf_{k \geq 0} \{(v_0, v_1) \in \mathcal{E}_k\}$. Then $\mathbf{P}(t_1 < C_1) \geq q$. We have

$$\begin{aligned} \left| \sum_{j \in \mathcal{N}_{v_1}(t_1)} a_{v_1 j}(t_1) x_j(t_1) - x_{v_0}(0) \right| &= |a_{v_1 v_0}(t_1) x_{v_0}(t_1) + a_{v_1 v_1}(t_1) x_{v_1}(t_1) - x_{v_0}(0)| \\ &= (1 - a_{v_1 v_0}(t_1)) |x_{v_1}(0) - x_{v_0}(0)| \\ &\leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|, \end{aligned} \quad (42)$$

which yields

$$\mathbf{P}\left(|x_{v_1}(t_1 + 1) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|\right) \geq P_{t_1}. \quad (43)$$

Thus, we obtain

$$\mathbf{P}\left(|x_{v_1}(C_1) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(0) - x_{v_0}(0)|\right) \geq \tilde{P}_0 q, \quad (44)$$

where $\tilde{P}_0 = \inf_{C_0 \leq \alpha < C_1} P_\alpha$. Repeating the analysis on time interval $[C_m, C_{m+1})$, $m = 1, 2, \dots$, we have

$$\mathbf{P}\left(|x_{v_1}(C_{m+1}) - x_{v_0}(0)| \leq (1 - \eta) |x_{v_1}(C_m) - x_{v_0}(0)|\right) \geq \tilde{P}_m q, \quad m = 1, 2, \dots \quad (45)$$

Similar to the proof of Proposition 2, connectivity independence leads to $\mathbf{P}(\lim_{k \rightarrow \infty} |x_{v_1}(k) - x_{v_0}(0)| = 0) = 1$. The claim is proved.

Further, we turn to $\bar{\mathcal{V}}_2$ and prove $\mathbf{P}(\lim_{k \rightarrow \infty} |x_l(k) - x_{v_0}(0)| = 0) = 1$, $l \in \bar{\mathcal{V}}_2$. Take $\varepsilon = 1/\ell$ for some $\ell \geq 1$. The previous claim implies that there exists $T(\ell) > 0$ such that $\mathbf{P}(|x_l(k) - x_{v_0}(0)| \leq 1/\ell, k \geq T, l \in \bar{\mathcal{V}}_1) = 1$. Suppose $C_{z_1} \geq T$ for some integer z_1 . Let v_2 be an arbitrary node in $\bar{\mathcal{V}}_2$. We next show $\mathbf{P}(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell) = 1$. There will be two cases.

- (i) If there exists $\varsigma_0 \geq 0$ such that $|x_{v_2}(\varsigma_0) - x_{v_0}(0)| \leq 1/\ell$, it holds that $|x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell$ for all $k \geq \varsigma_0$. Thus, denoting $E = \{|x_{v_2}(k) - x_{v_0}(0)| > 1/\ell \text{ for all } k \geq 0\}$ as an event, we have $\mathbf{P}(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell | E^c) = 1$.
- (ii) With probability one, all the neighbors of v_1 excluding itself in \mathcal{G}_k are within $\{v_0\} \cup \bar{\mathcal{V}}_1$ for all k , i.e., $\mathbf{P}(\mathcal{N}_{v_2}(k) \subseteq \{v_2\} \cup \{v_0\} \cup \bar{\mathcal{V}}_1, k = 0, 1, \dots) = 1$. Define $\hat{t}_1 = \inf_{k \geq C_{z_1}} \{\text{there is}$

at least one arc leaving from $\{v_0\} \cup \bar{\mathcal{V}}_1$ entering v_2 in \mathcal{E}_k . Then we obtain

$$\begin{aligned}
& \left| \sum_{j \in \mathcal{N}_{v_2}(\hat{t}_1)} a_{v_2 j}(\hat{t}_1) x_j(\hat{t}_1) - x_{v_0}(0) \right| \\
&= \sum_{j \in \{v_0\} \cup \bar{\mathcal{V}}_1} a_{v_2 j}(\hat{t}_1) |x_j(\hat{t}_1) - x_{v_0}(0)| + a_{v_2 v_2}(\hat{t}_1) |x_{v_2}(\hat{t}_1) - x_{v_0}(0)| \\
&\leq [1 - a_{v_2 v_2}(\hat{t}_1)] \frac{1}{\ell} + a_{v_2 v_2}(\hat{t}_1) |x_{v_2}(C_{\tau_1}) - x_{v_0}(0)| \\
&\leq [1 - \eta] \frac{1}{\ell} + \eta |x_{v_2}(C_{\tau_1}) - x_{v_0}(0)|,
\end{aligned} \tag{46}$$

which yields

$$\left| \sum_{j \in \mathcal{N}_{v_2}(\hat{t}_1)} a_{v_2 j}(\hat{t}_1) x_j(\hat{t}_1) - x_{v_0}(0) \right| - \frac{1}{\ell} \leq \eta \left[|x_{v_2}(C_{\tau_1}) - x_{v_0}(0)| - \frac{1}{\ell} \right]. \tag{47}$$

Consequently, denoting $y_m = |x_{v_2}(C_m) - x_{v_0}(0)| - 1/\ell$ for $m = 0, 1, \dots$, (47) leads to

$$\mathbf{P}\left(y_{z_1+1} \leq \eta \cdot y_{z_1}\right) \geq \tilde{P}_{z_1} q \tag{48}$$

and similarly

$$\mathbf{P}\left(y_{m+1} \leq \eta \cdot y_m\right) \geq \tilde{P}_m q, \quad m = z_1 + 1, \dots \tag{49}$$

When $\sum_{s=0}^{\infty} \tilde{P}_s = \infty$, based on the same argument as in the proof of Proposition 1, we see from (49) that $\lim_{m \rightarrow \infty} \mathbf{E}(y_m) = 0$, and therefore

$$\mathbf{P}\left(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell \mid \mathbf{E}\right) = 1. \tag{50}$$

As a result, we obtain $\mathbf{P}\left(\limsup_{k \rightarrow \infty} |x_{v_2}(k) - x_{v_0}(0)| \leq 1/\ell\right) = 1$. Since ℓ is chosen arbitrarily, we further conclude that

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} |x_l(k) - x_{v_0}(0)| = 0\right) = 1, \quad l \in \mathcal{V}_2.$$

Therefore, continuing the estimations on node set $\bar{\mathcal{V}}_3, \dots, \bar{\mathcal{V}}_{d_*}$, eventually we have $\mathbf{P}\left(\lim_{k \rightarrow \infty} |x_l(k) - x_{v_0}(0)| = 0\right) = 1$ for all $l \in \mathcal{V}$. The desired conclusion thus follows. \square

Proposition 3 leads to the following corollary.

Corollary 3 *Assume that $\mathbf{P}(\mathcal{G}([0, \infty))$ is acyclic) = 1.*

(i) *Suppose $\{\mathcal{G}_k\}_0^\infty$ is stochastically infinitely quasi-strongly connected and $P_{k+1} \leq P_k$ for all k . Then (1) achieves global a.s. consensus if $\sum_{m=0}^\infty P_{C_m} = \infty$.*

(ii) *Suppose $\{\mathcal{G}_k\}_0^\infty$ is stochastically uniformly quasi-strongly connected. Let either $B = 1$ or $P_{k+1} \leq P_k$, $k \geq 0$ hold. Then (1) achieves global a.s. consensus if and only if $\sum_{k=0}^\infty P_k = \infty$.*

4 Arc-independent Graphs

In this section, we consider arc-independent graphs and prove Theorem 4. We present the convergence analysis using a stochastic matrix argument.

Let $e_i = (0 \dots 1 \dots 0)^T$ be the $n \times 1$ unit vector with the i th component equal to 1. Denote $\bar{a}_i(k) = (\bar{a}_{i1} \dots \bar{a}_{in})^T$ as an $n \times 1$ unit vector with $\bar{a}_{ij}(k) = a_{ij}(k)$ if $j \in \mathcal{N}_i(k)$, and $\bar{a}_{ij}(k) = 0$ otherwise. Let $W(k) = (w_1(k) \dots w_n(k))^T \in \mathbb{R}^{n \times n}$ be a random matrix with

$$w_i(k) = \begin{cases} \bar{a}_i(k), & \text{with probability } P_k \\ e_i, & \text{with probability } 1 - P_k \end{cases} \quad (51)$$

for $i = 1, \dots, n$. Algorithm (1) is in a compact form:

$$x(k+1) = W(k)x(k). \quad (52)$$

We first establish several useful lemmas on the product of stochastic matrices, and then the proof for Theorem 4 is presented.

4.1 Stochastic Matrices

A finite square matrix $M = \{m_{ij}\} \in \mathbb{R}^{n \times n}$ is called *stochastic* if $m_{ij} \geq 0$ for all i, j and $\sum_j m_{ij} = 1$ for all i . For a stochastic matrix M , introduce

$$\delta(M) = \max_j \max_{\alpha, \beta} |m_{\alpha j} - m_{\beta j}|, \quad \lambda(M) = 1 - \min_{\alpha, \beta} \sum_j \min\{m_{\alpha j}, m_{\beta j}\}. \quad (53)$$

If $\lambda(M) < 1$ we call M a *scrambling* matrix. The following lemma can be found in [16].

Lemma 5 *For any k ($k \geq 1$) stochastic matrices M_1, \dots, M_k ,*

$$\delta(M_1 M_2 \dots M_k) \leq \prod_{i=1}^k \lambda(M_i). \quad (54)$$

Let $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ be a matrix with nonnegative entries. We can associate a unique digraph $\mathcal{G}_M = \{\mathcal{V}, \mathcal{E}_M\}$ with M on node set $\mathcal{V} = \{1, \dots, n\}$ such that $(j, i) \in \mathcal{E}_M$ if and only if $m_{ij} > 0$. We call \mathcal{G}_M the *induced graph* of M .

The following lemma is given on the induced graphs of products of stochastic matrices.

Lemma 6 *Let M_1, \dots, M_k be $k \geq 1$ stochastic matrices with positive diagonal entries. Then $(\bigcup_{i=1}^k \mathcal{G}_{M_i}) \subseteq \mathcal{G}_{M_k \dots M_1}$.*

Proof. We prove the case $k = 2$, and the conclusion will then follow by induction.

Denote $M_{ij}^{(1)}$, $M_{ij}^{(2)}$ and $M_{ij}^{(2)\langle 1 \rangle}$ as the ij -entries of M_1 , M_2 and M_2M_1 , respectively. Note that, we have

$$M_{ij}^{(2)\langle 1 \rangle} = \sum_{m=1}^n M_{im}^{(2)} M_{mj}^{(1)} \geq M_{ii}^{(2)} M_{ij}^{(1)} + M_{ij}^{(2)} M_{jj}^{(1)}. \quad (55)$$

The desired conclusion follows immediately from $M_{ii}^{(2)}, M_{jj}^{(1)} > 0$. \square

The following lemma helps in determine whether a product of stochastic matrices is a scrambling matrix.

Lemma 7 *Let M_1, \dots, M_{n-1} be $n-1$ stochastic matrices with positive diagonal entries. Suppose there exists a node $i_0 \in \mathcal{V}$ such that i_0 is a root of \mathcal{G}_{M_τ} for all $\tau = 1, \dots, n-1$. Then $M_{n-1} \cdots M_1$ is a scrambling matrix.*

Proof. We denote the ij -entry of M_s as $M_{ij}^{(s)}$ for $s = 1, \dots, n-1$. Since i_0 is a center of \mathcal{G}_{M_1} , at least one node i_1 exists different with i_0 such that $(i_0, i_1) \in \mathcal{E}_{M_1}$. This immediately implies $M_{i_1 i_0}^{(1)} > 0$ according to the definition of induced graph.

Further, we denote the ij -entry of M_2M_1 as $M_{ij}^{(2)\langle 1 \rangle}$. According to Lemma 6, we have $M_{i_1 i_0}^{(2)\langle 1 \rangle} > 0$ resulting from $M_{i_1 i_0}^{(1)} > 0$. Since i_0 is also a center of \mathcal{G}_{M_2} , there must be a node i_2 different with i_0 and i_1 such that there is at least one arc leaving from $\{i_0, i_1\}$ entering i_2 in \mathcal{G}_{M_2} . There will be two cases.

- (i) When $(i_0, i_2) \in \mathcal{E}_{M_2}$, we have $M_{i_2 i_0}^{(2)} > 0$, which implies $M_{i_2 i_0}^{(2)\langle 1 \rangle} > 0$ based on Lemma 6.
- (ii) When $(i_1, i_2) \in \mathcal{E}_{M_2}$, we have $M_{i_2 i_1}^{(2)} > 0$. Thus, we obtain

$$M_{i_2 i_0}^{(2)\langle 1 \rangle} = \sum_{\tau=1}^n M_{i_2 \tau}^{(2)} M_{\tau i_0}^{(1)} \geq M_{i_2 i_1}^{(2)} M_{i_1 i_0}^{(1)} > 0. \quad (56)$$

Similar analysis gives i_3, \dots, i_{n-1} such that $\mathcal{V} = \{i_0, \dots, i_{n-1}\}$. We eventually obtain

$$M_{i_m i_0}^{(n-1) \dots \langle 1 \rangle} > 0, \quad m = 0, 1, \dots, n-1 \quad (57)$$

where $M_{ij}^{(d_0) \dots \langle 1 \rangle}$ denotes the ij -entry of $M_{n-1} \cdots M_1$.

According to the definition of $\delta(M)$ in (53), (57) implies

$$\lambda(M_{n-1} \cdots M_1) \leq 1 - \min_{m=0, \dots, n-1} M_{i_m i_0}^{(n-1) \dots \langle 1 \rangle} < 1.$$

The desired conclusion follows. \square

4.2 Proof of Theorem 4: Convergence

This subsection presents the proof of the conclusion on a.s. consensus in Theorem 4. We only need to show the sufficiency part. Note that global a.s. consensus of (1) is equivalent with

$$\mathbf{P}\left(\lim_{k \rightarrow \infty} \delta(W(k) \cdots W(0)) = 0\right) = 1,$$

where $\delta(M)$ of a stochastic matrix M is defined in (53).

Node i *succeeds* at time k if it follows the averaging dynamics at time k , i.e., if $x_i(k+1) = \sum_{j \in \mathcal{N}_i(k)} a_{ij}(k)x_j(k)$. We define

$$\Psi_k = \begin{cases} 1, & \text{if at least one node succeeds at time } k; \\ 0, & \text{otherwise.} \end{cases} \quad (58)$$

Then, we have $\Psi_k = 1$ with probability $1 - (1 - P_k)^n$ and $\Psi_k = 0$ with probability $(1 - P_k)^n$. Moreover, Ψ_0, Ψ_1, \dots are independent. The following lemma is given on Ψ_k .

Lemma 8 $\mathbf{P}(\Psi_k = 1 \text{ for infinitely many } k) = 1$ if and only if $\sum_{k=0}^{\infty} P_k = \infty$.

Proof. Assume that $\sum_{k=0}^{\infty} P_k < \infty$. Lemma 1 implies that $\prod_{k=0}^{\infty} (1 - P_k) > 0$, which yields $\prod_{k=0}^{\infty} (1 - P_k)^n > 0$. Therefore, we have $\mathbf{P}(\Psi_k = 0, k = 0, \dots) > 0$. The necessity claim holds.

On the other hand, suppose $\sum_{k=0}^{\infty} P_k = \infty$. Then Lemma 1 leads to $\prod_{k=T}^{\infty} (1 - P_k)^n = 0$ for all $T \geq 0$. Thus,

$$\mathbf{P}(\Psi_k = 1 \text{ for finitely many } k) = \mathbf{P}(\exists T \geq 0, \text{ i.e., } \Psi_k = 0 \text{ for all } k \geq T) \leq \sum_{T=0}^{\infty} \prod_{k=T}^{\infty} (1 - P_k)^n = 0.$$

The sufficiency part therefore follows. \square

Noting the fact that

$$1 - ny \leq (1 - y)^n$$

for all $y \in [0, 1]$ and $n \geq 1$, we obtain

$$1 - (1 - P_k)^n \leq nP_k, \quad k = 0, \dots$$

Thus, we have

$$\mathbf{P}(\text{node } i \text{ succeeds at time } k \mid \Psi_k = 1) = \frac{P_k}{1 - (1 - P_k)^n} \geq \frac{P_k}{nP_k} = \frac{1}{n} \quad (59)$$

for all i and k .

Based on Lemma 8, with probability one, we can well define the (Bernoulli) sequence of $\{\Psi_k\}_0^\infty$,

$$\zeta_1 < \dots < \zeta_m < \zeta_{m+1} < \dots, \quad (60)$$

where ζ_m is the m 'th time when $\Psi_k = 1$. Then according to (59), for any $(i, j) \in \mathcal{E}^*$, we have

$$\mathbf{P}((i, j) \in \mathcal{G}_{W(\zeta_m)}) = \mathbf{P}(j \text{ succeeds at time } \zeta_m) \cdot \mathbf{P}((i, j) \in \mathcal{G}_{\zeta_m}) \geq \frac{\theta_0}{n} \quad (61)$$

for $m = 1, 2, \dots$.

Denote $Q_1 = W(\zeta_{|\mathcal{E}^*|}) \cdots W(\zeta_2)W(\zeta_1)$, where $|\mathcal{E}^*|$ represents the number of elements in \mathcal{E}^* . From (61), we can pick up (i_τ, j_τ) , $\tau = 1, \dots, |\mathcal{E}^*|$ as all the arcs in \mathcal{E}^* , and the arc-independence leads to

$$\mathbf{P}\left((i_\tau, j_\tau) \in \mathcal{G}_{W(\zeta_\tau)}, \tau = 1, \dots, |\mathcal{E}^*|\right) \geq \left(\frac{\theta_0}{n}\right)^{|\mathcal{E}^*|}, \quad (62)$$

which yields

$$\mathbf{P}\left(\mathcal{G}^* \subseteq \mathcal{G}_{Q_1}\right) \geq \mathbf{P}\left(\mathcal{G}^* \subseteq \left(\bigcup_{\tau=1}^{|\mathcal{E}^*|} \mathcal{G}_{W(\zeta_\tau)}\right)\right) \geq \left(\frac{\theta_0}{n}\right)^{|\mathcal{E}^*|} \quad (63)$$

according to Lemma 6.

We continue defining $Q_m = W(\zeta_{m|\mathcal{E}^*|}) \cdots W(\zeta_{(m-1)|\mathcal{E}^*|+1})$ for $m = 2, 3, \dots$, and similarly

$$\mathbf{P}\left(\mathcal{G}^* \subseteq \mathcal{G}_{Q_m}\right) \geq \left(\frac{\theta_0}{n}\right)^{|\mathcal{E}^*|} \quad (64)$$

for all m . Because \mathcal{G}^* is quasi-strongly connected, Lemma 7 yields

$$\mathbf{P}\left(\lambda(Q_{n-1} \cdots Q_1) < 1\right) \geq \left(\frac{\theta_0}{n}\right)^{(n-1)|\mathcal{E}^*|}. \quad (65)$$

Moreover, since $Q_{n-1} \cdots Q_1$ represents a product of $(n-1)|\mathcal{E}^*|$ stochastic matrices, each of which satisfies the weights rule. Therefore, each nonzero entry, \bar{Q}_{ij} of $Q_{n-1} \cdots Q_1$, satisfies

$$\bar{Q}_{ij} \geq \eta^{(n-1)|\mathcal{E}^*|}. \quad (66)$$

We see from (65) and (66) that

$$\mathbf{P}\left(\lambda(Q_{n-1} \cdots Q_1) \leq 1 - \eta^{(n-1)|\mathcal{E}^*|}\right) \geq \left(\frac{\theta_0}{n}\right)^{(n-1)|\mathcal{E}^*|}. \quad (67)$$

Denoting $U_m = Q_{m(n-1)} \cdots Q_{(m-1)(n-1)+1}$ for $m = 1, 2, \dots$, we can now further conclude

$$\mathbf{P}\left(\lambda(U_m) \leq 1 - \eta^{(n-1)|\mathcal{E}^*|}\right) \geq \left(\frac{\theta_0}{n}\right)^{(n-1)|\mathcal{E}^*|} \quad (68)$$

for all $m = 1, 2, \dots$.

Thus, based on Lemma 5, we have

$$\lim_{m \rightarrow \infty} \mathbf{E} \left[\delta(U_m \cdots U_1) \right] \leq \lim_{m \rightarrow \infty} \mathbf{E} \left[\prod_{\tau=1}^m \lambda(U_m) \right] = 0, \quad (69)$$

Then from Fatou's Lemma we have

$$\mathbf{P} \left(\lim_{m \rightarrow \infty} \delta(U_m \cdots U_1) = 0 \right) = 1, \quad (70)$$

which leads to

$$\mathbf{P} \left(\lim_{k \rightarrow \infty} \delta(W(k) \cdots W(0)) = 0 \right) = 1$$

since $W(k) = I_n$ for any $k \notin \{\zeta_1, \zeta_2, \dots\}$, where I_n is the identity matrix. Hence, we have obtained global a.s. consensus for (1).

4.3 Proof of Theorem 4: Computation Time

In this subsection, we establish the upper bound of $T_{\text{com}}(\epsilon)$ given in Theorem 4.

Denote the ij -entry of $W(k-1) \cdots W(0)$ as Φ_{ij} . Then for all i, j and τ , we have

$$\begin{aligned} |x_i(k) - x_j(k)| &= \left| \sum_{\alpha=1}^n \Phi_{i\alpha} x_\alpha(0) - \sum_{\alpha=1}^n \Phi_{j\alpha} x_\alpha(0) \right| \\ &= \left| \sum_{\alpha=1}^n \Phi_{i\alpha} (x_\alpha(0) - x_\tau(0)) - \sum_{\alpha=1}^n \Phi_{j\alpha} (x_\alpha(0) - x_\tau(0)) \right| \\ &\leq \sum_{\alpha=1}^n |\Phi_{i\alpha} - \Phi_{j\alpha}| \cdot \max_{\alpha} |x_\alpha(0) - x_\tau(0)| \\ &\leq n\delta(W(k-1) \cdots W(0)) \cdot \max_{\alpha} |x_\alpha(0) - x_\tau(0)|. \end{aligned} \quad (71)$$

Therefore, we obtain that for all $k \geq 1$,

$$\mathcal{H}(k) \leq n\delta(W(k-1) \cdots W(0)) \mathcal{H}(0). \quad (72)$$

Introduce a sequence of random variables

$$\xi_k = \max \{m : \zeta_m \leq k-1\} = \sum_{i=0}^{k-1} \Psi_i, \quad k = 0, 1, \dots$$

where ζ_m is the Bernoulli sequence of $\{\Psi_k\}_0^\infty$ defined in (60).

Denote $E_0 = (n-1)|\mathcal{E}^*|$. Then according to Lemma 5 and applying Markov's Inequality, (72) implies

$$\begin{aligned} \mathbf{P}\left(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\right) &\leq \mathbf{P}\left(\delta(W(k-1) \cdots W(0)) \geq \frac{\epsilon}{n}\right) \\ &\leq \frac{n}{\epsilon} \mathbf{E}\left[\delta(W(k-1) \cdots W(0))\right] \\ &\leq \frac{n}{\epsilon} \mathbf{E}\left[\lambda_{\lfloor \frac{\xi_k}{E_0} \rfloor} \cdots \lambda_1\right], \end{aligned} \quad (73)$$

where $E_0 = (n-1)|\mathcal{E}^*|$ and by definition $\lambda_m = \lambda(U_m)$ for $m = 1, 2, \dots$ and $\lfloor z \rfloor$ represents the largest integer no greater than z .

It is not hard to see from (65) that

$$\mathbf{E}[\lambda_m] \leq 1 - \left(\frac{\eta\theta_0}{n}\right)^{E_0}, \quad m = 1, 2, \dots, \quad (74)$$

which yields

$$\begin{aligned} \mathbf{E}\left[\lambda_{\lfloor \frac{\xi_k}{E_0} \rfloor} \cdots \lambda_1\right] &= \mathbf{E}\left[\mathbf{E}\left[\lambda_{\lfloor \frac{\xi_k}{E_0} \rfloor} \cdots \lambda_1 \mid \xi_k\right]\right] \\ &= \left(1 - \left(\frac{\eta\theta_0}{n}\right)^{E_0}\right)^{\mathbf{E}\left[\lfloor \frac{\xi_k}{E_0} \rfloor\right]} \\ &\leq \left(1 - \left(\frac{\eta\theta_0}{n}\right)^{E_0}\right)^{\sum_{i=0}^{k-1} \mathbf{E}[\Psi_i]/E_0 - 1} \\ &= \left(1 - \left(\frac{\eta\theta_0}{n}\right)^{E_0}\right)^{\sum_{i=0}^{k-1} [1 - (1-P_i)^n]/E_0 - 1} \end{aligned} \quad (75)$$

since the node dynamics is independent with the graph process.

Therefore, with (73) and (75), we have

$$\mathbf{P}\left(\frac{\mathcal{H}(k)}{\mathcal{H}(0)} \geq \epsilon\right) \leq \frac{n}{\epsilon} \left(1 - \left(\frac{\eta\theta_0}{n}\right)^{E_0}\right)^{\sum_{i=0}^{k-1} [1 - (1-P_i)^n]/E_0 - 1}, \quad (76)$$

so (5) can be obtained immediately by some simple algebra.

5 Conclusions

This paper investigated standard consensus algorithms coupled with randomized individual node decision-making over stochastically time-varying graphs. Each node determined its update by a sequence of Bernoulli trials with time-varying probabilities. The central aim of this work was to investigate the relation between the required level of independence of the graph process and the overall convergence. We consequently introduced connectivity-independence and arc-independence for graph processes. An impossibility theorem showed that a.s. consensus can

not be achieved unless the sum of the success probability sequence diverges. Then a series of sufficiency conditions were given for the network to reach a global a.s. consensus under various connectivity assumptions.

Particularly, when either the graph is arc-independent or acyclic, the sum of the success probability sequence diverging is a sharp threshold condition for consensus under a simple self-confidence assumption. In other words, consensus appears suddenly as the sum of the probability sequence goes to infinity. Consistent with classical random graph theory, this so-called zero-one law was to the best of our knowledge first established in this paper for distributed consensus computation over random graphs.

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